Local limit theorems for Markov chains associated with disk polynomials

Abstract

We prove local limit theorems for Markov chains on $\mathbb{N}_0^2$ associated with the disk polynomials of index $\alpha > 0$. This means that we study the rate of convergence of their transition probabilities. Our results complement the known central limit theorem for these Markov chains.

1 Introduction

The so called disk polynomials form a family of orthogonal polynomials in two variables on the unit disk. We use these polynomials to define a class of Markov chains with state space $\mathbb{N}_0^2$ which include as a special case isotropic random walks on the dual of the Gelfand pair $(U(d), U(d-1))$. These Markov chains have been studied by Bouhaik and Gallardo in an number of papers. In particular they established laws of large numbers and central limit theorems, see [2]–[4]. It is the purpose of this paper to supplement the central limit theorem with the corresponding local limit theorems. This yields information on the order of convergence of the transition probabilities. The proof uses the Hilb formula for the disk polynomials and an integral representation of the transition probabilities. This method of proof follows the pattern of the proof of the classical local limit theorems to be found e.g. in chapter 4 of [8]. Finally let us mention that the same method has also successfully been applied to Markov chains associated with certain one dimensional orthogonal polynomials including the Jacobi polynomials, see [5].

2 Preliminaries

The disk polynomials. Let $\alpha \geqslant 0$ and $(m, n) \in \mathbb{N}_0^2$. The function

(2.1) $$R_{m,n}^{(\alpha)}(z) = R_{m,n}^{(\alpha)}(re^{i\varphi}) := e^{i(m-n)\varphi} r^{|m-n|} P_{m\wedge n}^{(\alpha,|m-n|)}(2r^2 - 1),$$

where $P_n^{(\alpha,\beta)}(x)$ is the n-th Jacobi polynomial normalized by the requirement $P_n^{(\alpha,\beta)}(1) = 1$ for all $n \in \mathbb{N}$ and $m \wedge n = \min(m, n)$ is called the *disk polynomial of* *degree* (m,n) *and exponent* α. The disk polynomials form a family of orthogonal polynomials in two variables on the unit disk $D := \{z \in \mathbb{C} : |z| \leqslant 1\} = \{r\, e^{i\varphi} : 0 \leqslant r \leqslant 1, 0 \leqslant \varphi \leqslant 2\pi\}$ with respect to the measure

$$\lambda_\alpha = \frac{\alpha+1}{\pi}(1-x^2-y^2)^\alpha dxdy \qquad (z = x+iy).$$

These polynomials have been studied by several authors, see [3],[4] and [9]. In particular for $\alpha = d-2$ ($d \geqslant 3$ an integer) they appear in the expression for the spherical functions of the Gelfand pair $(U(d), U(d-1))$ (see e.g. [1], 3.1.14).

In [9] it is proven that all the linearization coefficients defined by

$$R^{(\alpha)}_{m_1,n_1}(z)R^{(\alpha)}_{m_2,n_2}(z) = \sum_{m,n} g(m_1,n_1,m_2,n_2,m,n)R^{(\alpha)}_{m,n}(z)$$

are nonnegative for $\alpha \geqslant 0$.

If we define a convolution of point measures on $\mathbb{N}_0^2$ by

$$\delta_{m_1,n_1} * \delta_{m_2,n_2} = \sum_{m,n} g(m_1,n_1,m_2,n_2,m,n)\delta_{m,n}$$

$\mathbb{N}_0^2$ becomes a commutative hypergroup with the involution $(m,n)^- = (n,m)$ and the neutral element $(0,0)$ and is called polynomial hypergroup in two variables on $\mathbb{N}_0^2$, compare [1], 3.1.4. For the general theory of hypergroups we refer to the monograph [1].

Our interest in this hypergroup structure stems from the fact that it allows a generalized harmonic analysis. In particular there exists a *Haar measure* m (i.e. a positive measure m satisfying $\delta_k * m = m$ for every $k \in \mathbb{N}_0^2$), which is uniquely determined by $m(\{0\}) = 1$. In our case it is given by

$$h_{k,l} := m(\{(k,l)\}) = \left(\int_D |R^{(\alpha)}_{k,l}(z)|^2 \lambda_\alpha(dz)\right)^{-1}.$$

Furthermore for any probability measure μ on $\mathbb{N}_0^2$ we can define a *(generalized) Fourier transform* $\hat{\mu}(z)$ as the continuous complex valued function

$$D \to \mathbb{C},\ \hat{\mu}(z) = \int_{\mathbb{N}_0^2} R^{(\alpha)}_{m,n}(z)\mu(\{(m,n)\}).$$

An important property of this Fourier transform is the convolution theorem $\widehat{\mu * \nu}(z) = \hat{\mu}(z)\cdot\hat{\nu}(z)$ if $z \in D$ ([1], Theorem 2.2.2 (a)).
Random walks. Let $\alpha \geqslant 0$ and $\mu \in M^1(\mathbb{N}_0^2)$. Every Markov chain on $\mathbb{N}_0^2$ with the transition kernel

$$P_{(i,j)(k,l)} = P(S_{n+1} = (k,l)|S_n = (i,j)) = \delta_{(i,j)} * \mu(\{(k,l)\})$$

is called *random walk (with law μ)* (note that this transition kernel is well defined by the positivity condition on the linearization coefficients). It is easy to see from the definition that the n-step transition probabilities are given by

$$P^{(n)}_{(i,j)(k,l)} := P(S_n = (k,l) \mid S_0 = (i,j)) = \delta_{(i,j)} * \mu^{(n)}(\{(k,l)\}) \tag{2.2}$$

where $\mu^{(n)}$ denotes the n-fold convolution product with respect to $*$. A thorough study of such Markov chains on an (arbitrary) hypergroup can be found in [1] or [7].

Remark. Choosing in particular $\mu = 1/2(\delta_{0,1} + \delta_{1,0})$ we obtain the Markov chain with transition kernel ([4], 2.16)

$$P_{(c,d)(k,l)} = \begin{cases} \frac{\alpha+c+1}{2(\alpha+c+d+1)} & \text{, if}(k,l) = (c+1,d) \\ \frac{c}{2(\alpha+c+d+1)} & \text{, if}(k,l) = (c-1,d) \\ \frac{\alpha+d+1}{2(\alpha+c+d+1)} & \text{, if}(k,l) = (c,d+1) \\ \frac{d}{2(\alpha+c+d+1)} & \text{, if}(k,l) = (c,d-1) \\ 0 & \text{otherwise.} \end{cases}$$

3 Local limit theorems

We start with some auxiliary results needed later. Throughout this section we assume that $\alpha > 0$.

Lemma 3.1. *Let $S_n = (X_n, Y_n)$ denote a random walk on $\mathbb{N}_0^2$ with law μ.*

(i) We have the following integral representation of the transition kernel

$$P^{(n)}_{(c,d)(k,l)} = h_{k,l} \int_D \hat{\mu}^n(z) \overline{R^{\alpha}_{c,d}(z)} R^{\alpha}_{k,l}(z) d\lambda_\alpha(z). \tag{3.1}$$

(ii) $\pi(S_n) = X_n - Y_n$ is an irreducible and aperiodic Markov chain on $\mathbb{Z}$.

(iii) $|\hat{\mu}(\cos t e^{i\varphi})| = 1 \Leftrightarrow t = 0 = \varphi$

Proof. (i) For $(c,d) \in \mathbb{N}_0^2$ fixed the Fourier transform $\widehat{\delta_{(c,d)} * \mu^{(n)}}(z)$ is integrable with respect to λ_α, which is the Plancherel measure on $\hat{\mathbb{N}_0^2}$. The inversion theorem [1], Theorem 2.2.36 yields the assertion.

(ii) S_n being irreducible, $\mathbb{N}_0^2$ is the smallest subhypergroup generated by $\operatorname{Tr}\mu$ ([7], Prop. 2.11).
$\pi(x,y) := x - y$ is a homomorphism of the hypergroups $\mathbb{N}_0^2$ and $\mathbb{Z}$ the latter endowed with its usual group structure ([4], Proposition 5.1) and $\pi(S_n)$ is a random walk on $\mathbb{Z}$ ([4], Theoreme 1). Thus $\mathbb{Z}$ is the smallest subgroup

generated by $\operatorname{Tr}\pi(\mu)$ and $\pi(S_n)$ is irreducible.
Now assume that $\pi(S_n)$ has period $d > 1$. Then $\pi(\mu^{(n)})(0) = 0$ for all n with $d \not| n$ and

$$0 = \pi(\mu^{(n)}(0)) = \mu^{(n)}(\pi^{-1}(0)) \geqslant \mu^{(n)}((0,0)) \geqslant 0.$$

This implies $P^{(n)}_{(0,0)(0,0)} = 0$ for all n with $d \not| n$ contradicting the aperiodicity of S_n.

(iii) We have

$$|R^{(\alpha)}_{m,n}(\cos te^{i\varphi})| = (\cos t)^{|m-n|}|R^{(\alpha,|m-n|)}_{m\wedge n}(\cos 2t)| < 1$$

for $0 < t \leqslant \pi/2$ and thus $|\hat{\mu}(\cos te^{i\varphi})| < 1$ for $t > 0$.
Furthermore

$$|\hat{\mu}(e^{i\varphi})| = |\sum_{(m,n)\in\mathbb{N}_0^2} \mu(\{(m,n)\})e^{i(m-n)\varphi}| = |\mathcal{F}(\pi(\mu))(\varphi)|,$$

where $\mathcal{F}$ is the usual Fourier transform on $\mathbb{Z}$. It follows from part (ii) that $\pi(S_n)$ is strongly aperiodic in the terminology of [10] (see [10], D 2.2 and D 5.1.). Thus $|\hat{\mu}(e^{i\varphi})| = 1 \Leftrightarrow \varphi = 0$ ([10], P 7.8). □

Moreover we need the following identities (compare [6], 7.7.3 (24) and (25))

$$\int_0^\infty e^{-Ct^2}\Lambda_\alpha(Nt)t^{2\alpha+1}dt = \frac{\Gamma(\alpha+1)}{2C^{\alpha+1}}\exp\left(-\frac{N^2}{4C}\right) \tag{3.2}$$

and

(3.3)
$$\int_0^\infty e^{-Ct^2}\Lambda_\alpha(Mt)\Lambda_\alpha(Nt)t^{2\alpha+1}dt = \frac{2^{2\alpha-1}\Gamma(\alpha+1)^2}{(NM)^\alpha C}\exp\left(-\frac{N^2+M^2}{4C}\right)I_\alpha\left(\frac{NM}{2C}\right).$$

Theorem 3.2. *Assume that the measure $\mu \in M^1(\mathbb{N}_0^2)$ satisfies the conditions*

(i) $\sum_{m,n}(m-n)\mu_{m,n} = 0$,

(ii) $\sum_{m,n}(m-n)^2\mu_{m,n} =: a < \infty$,

(iii) $\sum_{m,n}\left(\frac{2mn}{\alpha+1} + m + n\right)\mu_{m,n} =: b < \infty$,

(iv) the random walk with law μ is irreducible and aperiodic.

Then we have for $(k,l) \in \mathbb{N}_0^2$ fixed

$$\lim_{n\to\infty} \sup_{\substack{c\leqslant\sqrt{n}\\ d\leqslant\sqrt{n}}} \Big| n^{\alpha+3/2} P^{(n)}_{(c,d)(k,l)} - \frac{2^{2\alpha+1}\Gamma(\alpha)\Gamma(\alpha+1)}{\sqrt{2a\pi}\, b(\beta_{cd}\beta_{kl})^\alpha} h_{k,l} n^\alpha \times$$
$$\exp\left(-\frac{(d-c+k-l)^2}{2an} - \frac{\beta_{cd}^2+\beta_{kl}^2}{2bn}\right) I_\alpha\left(\frac{\beta_{cd}\beta_{kl}}{bn}\right)\Big| = 0$$

where $\beta_{kl} := \sqrt{(2k+\alpha+1)(2l+\alpha+1)}$.
In particular we have for both $(c,d)(k,l)$ and (k,l) fixed

$$\lim_{n\to\infty} n^{\alpha+3/2} P^{(n)}_{(c,d)(k,l)} = \frac{2^{\alpha+1/2}\Gamma(\alpha)}{\sqrt{\pi} b^{\alpha+1}} h_{k,l}.$$

Proof. The proof uses the Hilb formula for disk polynomials ([3], Theorem 1) and the integral representation (3.1). In addition we set

$$\hat{\mu}(t,\varphi) := \hat{\mu}(\cos t e^{i\varphi}) \qquad R_{c,d}(t,\varphi) := R^{(\alpha)}_{c,d}(\cos t e^{i\varphi})$$

and

$$f_n(t,\varphi) :=$$
$$\hat{\mu}(t/\sqrt{n},\varphi/\sqrt{n})^n R_{c,d}(t/\sqrt{n},\varphi/\sqrt{n}) \overline{R_{k,l}(t/\sqrt{n},\varphi/\sqrt{n})} \cos(t/\sqrt{n}) \sin^{2\alpha+1}(t/\sqrt{n}).$$

First we show that the asymptotics of $P^{(n)}_{(c,d)(k,l)}$ depends only on the integrand in a neighborhood of $(1,1)$. In a second step we determine the asymptotic behavior in this neighborhood.
To achieve this we split the integral representation as follows ($c,d,k,l \in \mathbb{N}_0^2$)

$$\begin{aligned}
n^{\alpha+3/2} P^{(n)}_{(c,d)(k,l)} &= \frac{n^{\alpha+1/2} h_{k,l}(\alpha+1)}{\pi} \left(\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \int_0^{\pi\sqrt{n}/2} f_n(t,\varphi)\,dt\,d\varphi \right) \\
&= \frac{n^{\alpha+1/2} h_{k,l}(\alpha+1)}{\pi} \left(\int_{-A}^{A} \int_0^{B} f_n(t,\varphi)\,dt\,d\varphi + \right. \\
&+ \left(\int_{-A}^{A} \int_B^{r\sqrt{n}} f_n(t,\varphi)\,dt\,d\varphi + \int_{[-s\sqrt{n},-A]\cup[A,s\sqrt{n}]} \int_0^{r\sqrt{n}} f_n(t,\varphi)\,dt\,d\varphi \right) \\
&+ \left.\left(\int_{-s\sqrt{n}}^{s\sqrt{n}} \int_{r\sqrt{n}}^{\pi\sqrt{n}/2} f_n(t,\varphi)\,dt\,d\varphi + \int_{[-\pi\sqrt{n},-s\sqrt{n}]\cup[s\sqrt{n},\pi\sqrt{n}]} \int_0^{\pi/2} f_n(t,\varphi)\,dt\,d\varphi \right)\right) \\
&= \frac{n^{\alpha+1/2} h_{k,l}(\alpha+1)}{\pi} \left(I_1(n) + I_2(n) + I_3(n) \right).
\end{aligned}$$

Choosing the constants A, B, r, s appropriately we can show that $n^{\alpha+1/2} I_2(n)$ and $n^{\alpha+1/2} I_3(n)$ tend to 0 uniformly in (c,d) as $n \to \infty$.

A Taylor expansion of $\hat{\mu}(t,\varphi)$ around $(0,0)$ shows that there exist $0<r<\pi/2$ and $0<s<\pi$ with $\hat{\mu}(t,\varphi)\leqslant 1-\frac{1}{4}(a\varphi^2+bt^2)$ for $0\leqslant t\leqslant r$ and $0\leqslant\varphi\leqslant s$. Furthermore for any $\varepsilon>0$ there exist $A>0$ and $B>0$ such that

$$\int_B^\infty e^{-bt^2/4}t^{2\alpha+1}dt<\varepsilon \qquad \text{and} \qquad \int_{[-A,A]^c} e^{-a\varphi^2/2}d\varphi<\varepsilon.$$

with this choice of A,B,r,s we have

$$\hat{\mu}(t/\sqrt{n},\varphi/\sqrt{n})^n\leqslant\left(1-\frac{1}{4n}(bt^2+a\varphi^2)\right)^n\leqslant e^{-bt^2/4}e^{-a\varphi^2/4}.$$

on the set $[B,r\sqrt{n}]\times[-A,A]\cup[0,r\sqrt{n}]\times\{[-s\sqrt{n},-A]\cup[A,s\sqrt{n}]\}$ and consequently

$$\begin{aligned}|n^{\alpha+1/2}I_2(n)| &\leqslant n^{\alpha+1/2}\Big(\int_{-A}^A e^{-a\varphi^2/4}d\varphi\int_B^\infty e^{-bt^2/4}(t/\sqrt{n})^{2\alpha+1}dt+\\ &\qquad \int_{[-A,A]^c} e^{-a\varphi^2/4}d\varphi\int_0^\infty e^{-bt^2/4}(t/\sqrt{n})^{2\alpha+1}dt\Big)\\ &\leqslant C\varepsilon\end{aligned}$$

uniformly in (c,d).
Using Lemma 3.1 we can find $\delta>0$ with $|\hat{\mu}(t,\varphi)|\leqslant 1-\delta$. if $r<t\leqslant\pi/2$ or $s<|\varphi|\leqslant\pi$. This yields

$$\begin{aligned}|n^{\alpha+1/2}I_3(n)|\leqslant n^{\alpha+3/2}\Big(&\int_{-s}^s\int_r^{\pi/2}|\hat{\mu}(t,\varphi)|^n\sin^{2\alpha+1}tdtd\varphi+\\ &\int_{s\leqslant|\varphi|\leqslant\pi}\int_0^{\pi/2}|\hat{\mu}(t,\varphi)|^n\sin^{2\alpha+1}tdtd\varphi\Big)\\ &\leqslant C(1-\delta)^n n^{\alpha+3/2}.\end{aligned}$$

Thus we obtain $|I_3(n)|\to 0$ uniformly in (c,d) as $n\to\infty$.
To finish the proof we show that $I_1(n)$ has the correct asymptotic behavior. In order to do this we write $I_1(n)$ as

$$\begin{aligned}&n^{\alpha+1/2}\int_{-A}^A\int_0^B f_n(t,\varphi)dtd\varphi=\\ &\int_{-\infty}^\infty e^{-a\varphi^2/2}e^{i\varphi/\sqrt{n}(d-c+k-l)}d\varphi\int_0^\infty e^{-bt^2/2}\Lambda_\alpha(\beta_{c,d}t/\sqrt{n})\Lambda_\alpha(\beta_{k,l}t/\sqrt{n})t^{2\alpha+1}dt\\ &+J_1(n)+J_2(n)+J_3(n)+J_4(n)\\ &=\frac{2^{2\alpha+1/2}\Gamma(\alpha)\Gamma(\alpha+1)}{\sqrt{a\pi}b(\beta_{cd}\beta_{kl})^\alpha}n^\alpha e^{-(d-c+k-l)^2/(2\alpha n)}e^{-\frac{\beta_{cd}^2+\beta_{kl}^2}{2nb}}I_\alpha\left(\frac{\beta_{cd}\beta_{kl}}{2bn}\right)\\ &\quad+J_1(n)+J_2(n)+J_3(n)+J_4(n),\end{aligned}$$

where

$$J_1(n) =$$
$$\int_{-A}^{A}\int_0^B \left[\hat\mu(t/\sqrt{n},\varphi/\sqrt{n})^n - e^{-a/2\varphi^2}e^{b/2t^2}\right] e^{i\varphi/\sqrt{n}(d-c+k-l)} \times$$
$$\Lambda_\alpha(\beta_{c,d}t/\sqrt{n})\Lambda_\alpha(\beta_{k,l}t/\sqrt{n})t^{2\alpha+1}dtd\varphi$$
$$J_2(n) =$$
$$\int_{-A}^{A}\int_0^B \hat\mu(t/\sqrt{n},\varphi/\sqrt{n})^n e^{i\varphi/\sqrt{n}(d-c+k-l)} \times$$
$$\left[\left(\frac{\sin t/\sqrt{n}}{t/\sqrt{n}}\right)^{2\alpha+1} R_{c,d}(t/\sqrt{n},0)R_{k,l}(t/\sqrt{n},0)\cos t/\sqrt{n} - \Lambda_\alpha(\beta_{c,d}t/\sqrt{n})\Lambda_\alpha(\beta_{k,l}t/\sqrt{n})\right] dtd\varphi$$
$$J_3(n) =$$
$$\int_{[-A,A]^c}\int_0^B e^{i\varphi/\sqrt{n}(d-c+k-l)-a\varphi^2/2-bt^2/2}\Lambda_\alpha(\beta_{c,d}t/\sqrt{n})\Lambda_\alpha(\beta_{k,l}t/\sqrt{n})t^{2\alpha+1}dtd\varphi$$
$$J_4(n) =$$
$$\int_{-\infty}^{\infty}\int_B^{\infty} e^{i\varphi/\sqrt{n}(d-c+k-l)-a\varphi^2/2-bt^2/2}\Lambda_\alpha(\beta_{c,d}t/\sqrt{n})\Lambda_\alpha(\beta_{k,l}t/\sqrt{n})t^{2\alpha+1}dtd\varphi.$$

Here we have used equation (3.3) and the identity

$$R_{c,d}(t,\varphi) = e^{i(c-d)\varphi}R_{c,d}(t,0)$$

(see 2.1).
It remains to be shown that $J_i(n) \to 0$ as $n \to \infty$ $(i = 1, 2, 3, 4)$.
This is immediate for $J_3(n)$ and $J_4(n)$.
Since $\hat\mu(t/\sqrt{n},\varphi/\sqrt{n})^n \to e^{-a\varphi^2/2}e^{-bt^2/2}$ uniformly on compact sets, $J_1(n) \to 0$ as $n \to \infty$. Regarding $J_2(n)$ set

$$f_n(t) :=$$
$$\left(\frac{\sin t/\sqrt{n}}{t/\sqrt{n}}\right)^{2\alpha+1} R_{c,d}(t/\sqrt{n},0)R_{k,l}(t/\sqrt{n},0)\cos t/\sqrt{n} - \Lambda_\alpha(\beta_{c,d}t/\sqrt{n})\Lambda_\alpha(\beta_{k,l}t/\sqrt{n}).$$

Now we use the Hilb formula for the disk polynomials ([3], Theorem 1) to show that

$$\lim_{n\to\infty} \sup_{\substack{c\leqslant\sqrt{n}\\ d\leqslant\sqrt{n}}} |f_n(t)| = 0$$

uniformly on compact sets.
This is obvious if $c = d = k = l = 0$.
The Hilb formula yields constants C_1–C_4 such that

$$|f_n(t)| \leqslant C_1t^2/n+C_2((c-d)^2+(k-l)^2)t^4/n^2+C_3t^4/n^2+C_4((c-d)^2+(k-l)^2)t^8/n^4,$$

if $cdkl > 0$. In the remaining cases we obtain analogous inequalities and the proof is complete. □

The above mentioned random walk with law $\mu = 1/2(\delta_{(1,0)} + \delta_{(0,1)}$ has period 2. Thus theorem 3.2 is not directly applicable. Nevertheless we have

Corollary 3.3. *Let S_n be the random walk with law $\mu = 1/2(\delta_{(1,0)} + \delta_{(0,1)}$ and assume that $c-k$ and $d-l$ are both even or odd. Then*

$$\lim_{n\to\infty} \Big| (2n)^{\alpha+3/2} P^{(2n)}_{(c,d)(k,l)} - \frac{2^{2\alpha+2}\Gamma(\alpha+2)\Gamma(\alpha+1)}{\sqrt{2\pi}(\beta_{cd}\beta_{kl})^\alpha} h_{k,l} n^\alpha \times \exp\left(-\frac{(d-c+k-l)^2}{n} - \frac{\beta_{cd}^2+\beta_{kl}^2}{2n}\right) I_\alpha\left(\frac{\beta_{cd}\beta_{kl}}{n}\right)\Big| = 0$$

where $\beta_{cd} = \sqrt{(2c+\alpha+1)(2d+\alpha+1)}$.

Proof. Write

$$\begin{aligned}(2n)^{\alpha+3/2} P^{(2n)}_{(c,d)(k,l)} &= 4(\alpha+1)h_{k,l}\left(\frac{\sqrt{2n}}{4\pi}\int_{-\pi}^{\pi}(\cos x)^{2n} e^{i(d-c+k-l)x}dx\right) \\ &\quad \left((2n)^{2\alpha+1}\int_0^{\pi/2}(\cos y)^{2n+1}(\sin y)^{2\alpha+1}R^{(\alpha)}_{c,d}(\cos y)R^{(\alpha)}_{k,l}(\cos y)dy\right) \\ &= 4(\alpha+1)h_{k,l}I_1(n)I_2(n).\end{aligned}$$

Furthermore

$$I_1(n) = \frac{\sqrt{2n}}{2} P(\sum_{k=1}^n X_k = d-c+k-l),$$

where X_k are independent identically distributed random variables taking values in $\mathbb{Z}$ and common law $\mu = 1/4\delta_{-2} + 1/2\delta_0 + 1/4\delta_2$. The classical local limit theorem ([8], Theorem 4.2.1) implies

$$\lim_{n\to\infty}\Big| I_1(n) - \frac{1}{\sqrt{2\pi}}\exp(-(d-c+k-l)^2/n)\Big| = 0$$

uniformly in c,d,k,l.
As in the proof above we may show that

$$\lim_{n\to\infty}\Big| I_2(n) - \frac{2^{2\alpha}\Gamma(\alpha+1)^2}{(\beta_{cd}\beta_{kl})^\alpha} n^\alpha \exp\left(-\frac{\beta_{cd}^2+\beta_{kl}^2}{2n}\right) I_\alpha\left(\frac{\beta_{cd}\beta_{kl}}{n}\right)\Big| = 0.$$

□

The following corollary justifies the term local limit theorem, since $\Phi_{a,b}(x,y)$ is the density of the limit law in the central limit theorem.

Corollary 3.4. *Under the assumptions of theorem 3.2 let k_n, l_n be sequences of natural numbers satisfying $k_n \to \infty$, $l_n \to \infty$ and $k_n, l_n = O(\sqrt{n})$. Then*

$$P^{(n)}_{(0,0)(k_n,l_n)} \approx \frac{1}{n}\Phi_{a,b}\left(\frac{k_n}{\sqrt{n}}, \frac{l_n}{\sqrt{n}}\right)$$

where

$$\Phi_{a,b}(x,y) = \frac{2^{\alpha+1}}{\sqrt{2\pi a}b^{\alpha+1}\Gamma(\alpha+1)}(xy)^\alpha(x+y)e^{-2xy/b}e^{-\frac{(x-y)^2}{2a}}.$$

Proof. We have with the same abbreviations as above

$$\frac{nP^{(n)}_{(0,0)(k_n,l_n)}}{\Phi_{a,b}(\frac{k_n}{\sqrt{n}}, \frac{k_n}{\sqrt{n}})} = \frac{\sqrt{2a}b^{\alpha+1}\Gamma(\alpha+2)h_{k_n,l_n}}{\sqrt{\pi}2^{\alpha+1}(k_n,l_n)^\alpha(k_n+l_n)}n^{\alpha+1/2}e^{\frac{2k_nl_n}{2bn}}e^{\frac{(k_n-l_n)^2}{2an}} \times$$
$$\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}}\int_0^{\pi\sqrt{n}/2} \hat{\mu}(t/\sqrt{n},\varphi/\sqrt{n})^n R_{k_n,l_n}(t/\sqrt{n},\varphi/\sqrt{n})\cos t/\sqrt{n}\sin^{2\alpha+1}t/\sqrt{n}dtd\varphi.$$

As

$$h_{k_n,l_n} = \frac{(k_n+l_n+\alpha+1)\Gamma(k_n+\alpha+1)\Gamma(l_n+\alpha+1)}{k_n!l_n!\Gamma(\alpha+1)\Gamma(\alpha+2)}$$

([4],(2.3)) the asymptotic of the Gamma function yields

$$\lim_{n\to\infty}\frac{\sqrt{2a}b^{\alpha+1}\Gamma(\alpha+2)h_{k_n,l_n}}{\sqrt{\pi}2^{\alpha+1}(k_n,l_n)^\alpha(k_n+l_n)}n^{\alpha+1/2} = \frac{\sqrt{2a}b^{\alpha+1}}{\sqrt{\pi}2^{\alpha+1}\Gamma(\alpha+1)}.$$

Thus all we have to show is

$$\lim_{n\to\infty}\frac{\sqrt{2a}b^{\alpha+1}}{\sqrt{\pi}2^{\alpha+1}\Gamma(\alpha+1)}e^{\frac{2k_nl_n}{2bn}}e^{\frac{(k_n-l_n)^2}{2an}}\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}}\int_0^{\pi\sqrt{n}/2}\hat{\mu}(t/\sqrt{n},\varphi/\sqrt{n})^n \times$$
$$R_{k_n,l_n}(t/\sqrt{n},\varphi/\sqrt{n})\cos t/\sqrt{n}\sin^{2\alpha+1}t/\sqrt{n}dtd\varphi = 1.$$

But this can be seen as in the proof of theorem 3.2 using equation (3.2) and $k_n, l_n = O(\sqrt{n})$. □

References

[1] Bloom, W. R., Heyer, H.: Harmonic Analysis of Probability Measures on Hypergroups. Berlin: DeGruyter 1994

[2] Bouhaik, M., Gallardo, L.: Une loi des grandes nombres et un théorème limite central pour les chaines de Markov sur $\mathbb{N}_0^2$ associées aux polynômes discaux. C.R. Acad. Sci. Paris Séries I Math. **310** no. 10, 739–744 (1990)

[3] Bouhaik, M., Gallardo, L.: A Mehler-Heine formula for disk polynomials. Indag. Math. N.S. **1**, 9–18 (1991)

[4] Bouhaik, M., Gallardo, L.: Un théorème limite central dans un hypergroupe bidimmensionnel. Ann. Inst. H. Poincare (2) **28** 47–61 (1992)

[5] Ehring, M.: Local limit theorems for Markov chains on a class of polynomial hypergroups. Preprint.

[6] Erdelyi, A. et al.: Higher Transcendental Functions. New York: McGraw Hill 1953

[7] Gallardo, L., Gebuhrer, O.: Marches alèatoires et hypergroupes. Expo. Math. **5**, 41–73 (1987)

[8] Ibragimov, I.A., Linnik, Y.V.: Independent and Stationary Sequences of Random Variables. Groningen: Wolters-Noordhoff Publishing 1971

[9] Koornwinder, T.: Positivity proofs for linearization and connection coefficients of orthogonal polynomials satisfying an addition formula. J. London Math. Soc. **28**, 101-114 (1978)

[10] Spitzer, F.: Principles of random walk. Princeton: Van Nostrand 1964